

On the Probability of Error for Triangular Quadrature Amplitude Modulation

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Abstract

We compute the exact value of error probability per symbol (SER) for triangular quadrature amplitude modulation (TQAM) scheme in the case of AWGN channel. The results show that the exact value of SER follows the behavior of the known upper bound [2]. Hence a simple modification of the upper bound can be used in practice for evaluating the SER.

Keywords: TQAM, symbol error probability, AWGN channel

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1 Introduction

One of the most popular modulation in commercial communication systems is square quadrature amplitude modulation (SQAM). SQAM scheme with its simple detection procedure is easy for implementation and demonstrates a good performance.

Recently, the triangular quadrature amplitude modulation (TQAM) was proposed. In TQAM constellation the signal points are vertexes of a lattice of equilateral triangles and the constellation is symmetric with respect to the origin. The comparison of TQAM with SQAM given in [3] shows that the former is more power efficient while preserves the low detection complexity of the latter. In [4] a general formula for calculating the average energy per symbol of the TQAM is derived and approximate values of symbol error rate (SER) and bit error rate (BER) of the TQAM in the presence of additive white Gaussian noise (AWGN) are given.

In the next section we give a brief description of TQAM. In Section 3 we derived the exact value of SER for 2^{2m} -TQAM constellations, uncoded case. In the last section we compare the obtained results with the upper bound given in [2].

2 TQAM constellation

In this paper we consider TQAM constellation of $M = 2^{2m}$ signal points placed in $L = 2^m$ rows parallel to real axis with L signal point in each row. The points form a lattice of equilateral triangles and the constellation is symmetric with respect to the origin. An example of 64-ary TQAM is given in Fig. 1.

The power gain of M-ary TQAM over M-ary SQAM in decibels [4] is

$$10 \log_{10} \left(\frac{8M - 8}{7M - 4} \right) \xrightarrow{M \rightarrow \infty} 0.5799 \text{ dB}$$

3 The SER in uncoded case

The L^2 -TQAM constellation can be separated into seven types of detection regions D_1, D_2, \dots, D_7 . In this section we will calculate the probability of correct detection q_i for each of the regions D_i , $i = 1, \dots, 7$. The number of detection regions of each type for L^2 -TQAM is given in the next table.

D_1	D_2	D_3	D_4	D_5	D_6	D_7
$(L-2)^2$	$2(L-2)$	$L-2$	2	2	2	$L-4$

Note that in the case of 16-TQAM ($L = 4$) the region D_7 does not exist.

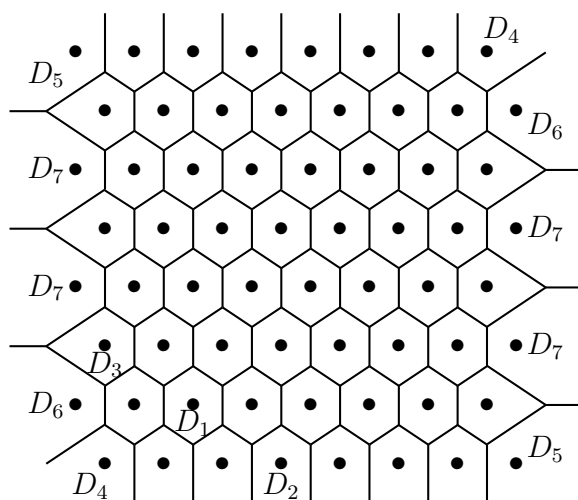


Figure 1. 64-TQAM constellation.

The lines that are boundaries of any region D_i have the following equations according to the coordinate system with the signal point in the region D_i as origin:

- $y = \frac{2d+x}{\sqrt{3}}$ the line left-above; $y = \frac{2d-x}{\sqrt{3}}$ the line right-above;
- $y = \frac{-x-2d}{\sqrt{3}}$ the line left-down; $y = \frac{x-2d}{\sqrt{3}}$ the line right-down;
- $x = \text{const}$ vertical line; $y = \text{const}$ horizontal line.

3.1 Region D_1 (hexagonal)

It consists of 4 congruent subregions. The right-above one is defined by $D_1 : \left\{ 0 \leq x \leq d; 0 \leq y \leq \frac{2d-x}{\sqrt{3}} \right\}$. Recalling that $\Pr \left\{ 0 \leq y \leq \frac{2d-x}{\sqrt{3}} \right\} = \frac{1}{2} \text{erf} \left(\frac{2d-x}{\sqrt{3}} \right)$ (e.g. see [1]) we conclude that

$$q_1 = \frac{2}{\sqrt{\pi N_0}} \int_0^d e^{-x^2/N_0} \text{erf} \left(\frac{2d-x}{\sqrt{3N_0}} \right) dx \quad (1)$$

3.2 Region D_2

These regions are symmetrically placed at the top and bottom of the constellation. Each of them is one side unbounded in y . Let us consider the region with negative values of y . It consists of two congruent parts and the right part is defined by $\left\{0 \leq x \leq d; \quad -\infty \leq y \leq \frac{2d-x}{\sqrt{3}}\right\}$. Obviously, it can be separated into two parts: rectangular $\{0 \leq x \leq d; \quad -\infty \leq y \leq 0\}$ and one forth of D_1 . Thus $q_2/2 = q_1/4 + (1/2)\text{erf}(d/\sqrt{N_0}) \cdot (1/2)$ or

$$q_2 = \frac{1}{2}q_1 + \frac{1}{2}\text{erf}\left(\frac{d}{\sqrt{N_0}}\right) \quad (2)$$

3.3 Region D_3 (pentagonal)

The probability of correct detection is

$$q_3 = \frac{1}{2}q_1 + 2 \Pr \left\{ -2d \leq x \leq 0; \quad 0 \leq y \leq \frac{2d+x}{\sqrt{3}} \right\}$$

Changing $x \rightarrow -x$ we get

$$q_3 = \frac{1}{2}q_1 + \frac{1}{\sqrt{\pi N_0}} \int_0^{2d} e^{-\frac{x^2}{N_0}} \text{erf}\left(\frac{2d-x}{\sqrt{3N_0}}\right) dx \quad (3)$$

3.4 Region D_4

This is infinite region that is union of D_2 and infinite triangle bounded by the lines $x = -d$ and $y\sqrt{3} = 2d + x$.

Hence changing $x \rightarrow -x$ and using the definition of error function we have

$$q_4 = q_2 + \frac{1}{4} \left(1 - \text{erf}\left(\frac{d}{\sqrt{N_0}}\right) \right) + \frac{1}{2\sqrt{\pi N_0}} \int_d^\infty e^{-\frac{x^2}{N_0}} \text{erf}\left(\frac{2d-x}{\sqrt{3N_0}}\right) dx \quad (4)$$

3.5 Region D_5

D_5 is infinite rectangular $\{-\infty \leq x \leq d, \quad -d\sqrt{3} \leq y \leq \infty\}$ with cut angle, the triangle $\Delta : \{-d \leq x \leq d, \quad -d\sqrt{3} \leq y \leq (x-2d)/\sqrt{3}\}$. Therefore

$$\begin{aligned} q_5 = & \frac{1}{4} \left(1 + \text{erf}\left(\frac{d}{\sqrt{N_0}}\right) + \text{erf}\left(\frac{d\sqrt{3}}{\sqrt{N_0}}\right) \right) - \frac{1}{4} \text{erf}\left(\frac{d\sqrt{3}}{\sqrt{N_0}}\right) \text{erf}\left(\frac{d}{\sqrt{N_0}}\right) \\ & + \frac{1}{2\sqrt{\pi N_0}} \int_{-d}^d e^{-\frac{x^2}{N_0}} \text{erf}\left(\frac{2d-x}{\sqrt{3N_0}}\right) dx \end{aligned} \quad (5)$$

3.6 Region D_6

D_6 can be obtained from D_5 by subtraction of the infinite triangle $\delta : \{-\infty \leq x \leq d, -\infty \leq y \leq \frac{x-2d}{\sqrt{3}}\}$, i.e., $q_6 = q_5 - q_\delta$.

Therefore

$$q_6 = q_5 - \frac{1}{4} \left(1 + \operatorname{erf} \left(\frac{d}{\sqrt{N_0}} \right) \right) + \frac{1}{2\sqrt{\pi N_0}} \int_{-\infty}^d e^{-\frac{x^2}{N_0}} \operatorname{erf} \left(\frac{2d-x}{\sqrt{3N_0}} \right) dx \quad (6)$$

3.7 Region D_7

This region exists only for $L \geq 6$. It is an infinite rectangular with cut two angles: the symmetrically placed triangles τ and τ' , i.e.,

$$D_7 : \{-\infty \leq x \leq d, -d\sqrt{3} \leq y \leq d\sqrt{3}\} \setminus \{\tau \cup \tau'\}, \quad \text{where}$$

$$\tau : \{-d \leq x \leq d, \frac{x-2d}{\sqrt{3}} \leq y \leq d\sqrt{3}\}; \quad \tau' : \{-d \leq x \leq d, -d\sqrt{3} \leq y \leq \frac{x-2d}{\sqrt{3}}\}$$

Since τ and τ' have the same contribution to the probability and τ' coincides with Δ from Subsection E we can write

$$q_7 = \frac{1}{2} \left(1 - \operatorname{erf} \left(\frac{d}{\sqrt{N_0}} \right) \right) \operatorname{erf} \left(\frac{d\sqrt{3}}{\sqrt{N_0}} \right) + \frac{1}{\sqrt{\pi N_0}} \int_{-d}^d e^{-\frac{x^2}{N_0}} \operatorname{erf} \left(\frac{2d-x}{\sqrt{3N_0}} \right) dx \quad (7)$$

Hence the probability q for a correct detection of the received signal point is

$$q = \frac{1}{L^2} [(L-2)^2 q_1 + 2(L-2)q_2 + (L-2)q_3 + 2(q_4 + q_5 + q_6) + (L-4)q_7]$$

and thus

$$SER = 1 - q. \quad (8)$$

4 Conclusion

Calculation of SER by the formulas given in the previous section requires the use of a power mathematical software which is not very practical. Fortunately we observed that the upper bound for the uncoded case derived in [2] has a behavior very similar to that of the exact value of SER versus signal/noise ratio in dB. This was the start point for us to find the following approximations.

$$16\text{TQAM}: \quad SER \approx e^{-\frac{12.75}{108} SNR} \left(\frac{|SNR_{dB} - 16|}{100} + 0.645 \right) \quad (9)$$

$$64\text{TQAM:} \quad \text{SER} \approx e^{-\frac{12.70}{444} \text{SNR}} \left(\frac{|\text{SNR}_{dB} - 20|}{100} + 0.79 \right) \quad (10)$$

$$256\text{TQAM:} \quad \text{SER} \approx e^{-\frac{12.65}{1788} \text{SNR}} \left(\frac{|\text{SNR}_{dB} - 28|}{100} + 0.83 \right) \quad (11)$$

where SNR_{dB} is the signal/noise ratio in decibels and $\text{SNR} = 10^{\frac{\text{SNR}_{dB}}{10}}$.

Figure 2 demonstrates how good is the approximation for 256-TQAM.

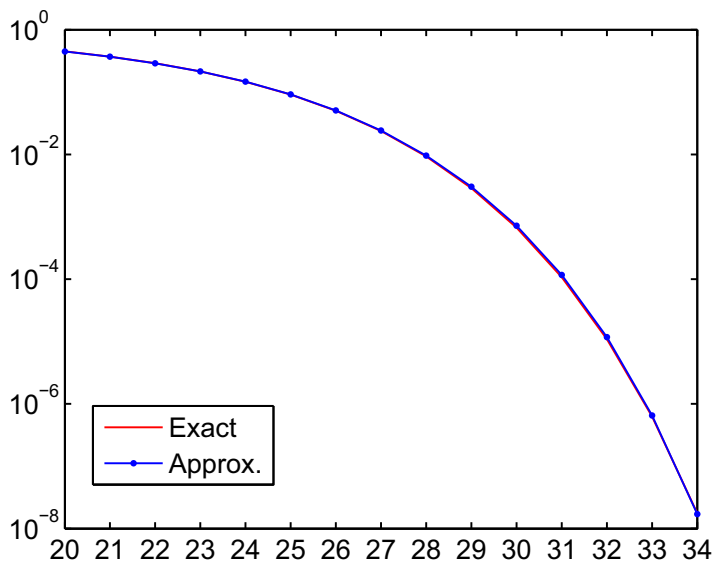


Figure 2. SER and approximation for 256-TQAM.

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